

Interaction of a quantum field with a rotating heat bath

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Abstract

The linear coupling of a rotating heat bath to a quantum field is studied in the framework of the Markovian master equation for the field's non-unitary time evolution. The bath's rotation induces population inversion for the field's low-energy modes. For bosons, this leads to superradiance, an irreversible process in which some of the bath's kinetic energy is extracted by spontaneous and stimulated emission. We find the energy and entropy balance for such systems and apply our results to the theory of black hole radiation. We also comment on how this relates to classical self-oscillations such as the hydrodynamic Kelvin-Helmholtz instability.

1 Introduction

Zel'dovich predicted in 1971 that a rotating black hole (BH) would radiate [1, 2]. His reasoning was based on the observation that the same physics that causes damping of an incident electromagnetic field by a static dielectric implies that, if the dielectric's surface moves faster than the incident field's phase velocity, then the incident field will be amplified at the expense of the dielectric's kinetic energy. Amplification of radiation by the sort of process that Zel'dovich described is often called “superradiance”, a term introduced by Misner in 1972 [3]. It is an irreversible process, distinct from the equilibrium phenomenon, also

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identified as “superradiance”, first described by Dicke in [4] and about which we will have nothing further to say here.

Zel’dovich’s thermodynamic argument was notably refined by Bekenstein and Schiffer in [5]. More recently, the irreversibility of electromagnetically superradiant systems has been carefully investigated in [6, 7]. For a thorough, modern review of rotational superradiance and its applications (with an emphasis on gravitational physics), see [8].

Zel’dovich’s argument applies to any object capable of damping a classical or bosonic degree of freedom. Some instances are Cherenkov radiation [9, 10], sonic booms and Mach shocks [11], the Landau criterion for superflows [12], the Moon’s tidal acceleration [13], the hydrodynamic Kelvin-Helmholtz instability by which the wind makes waves on the ocean’s surface [14], and mechanical instabilities of rotors.¹ The prediction of BH superradiance motivated Hawking’s subsequent discovery that a static BH must radiate thermally [16].²

The novelty of our approach here is to provide a general and self-contained treatment of superradiance, based on the linear coupling of a quantum field to a rotating heat bath. Our computations rely on the formalism of the Markovian master equation (also called the “Lindblad equation”, after one of its developers) for an open quantum system [19, 20, 21]. Our results help illuminate the necessary connection between superradiance and the thermal dynamics of the bath, which for a stationary BH lead to Hawking radiation. We also clarify the connection between superradiance of bosonic quantum fields and classical self-oscillations, using as example a flow instability.

2 Model and approximations

Consider a quantum field, either bosonic or fermionic, interacting with a source that acts as an equilibrium heat bath. Let the heat bath rotate with angular velocity Ω about its symmetry axis, which we take to be the z -axis. The free quantum field is described by the set of annihilation and creation operators $a_{m\alpha}(\omega)$, $a_{m\alpha}^\dagger(\omega)$, corresponding to the field modes $|\omega, m, \alpha\rangle$ and satisfying the (anti-)commutation relations

$$\begin{aligned} [a_{m\alpha}(\omega), a_{m'\alpha'}^\dagger(\omega')]_{\pm} &= \delta_{\omega\omega'} \delta_{mm'} \delta_{\alpha\alpha'} , \\ [a_{m\alpha}(\omega), a_{m'\alpha'}(\omega')]_{\pm} &= [a_{m\alpha}^\dagger(\omega), a_{m'\alpha'}^\dagger(\omega')]_{\pm} = 0 . \end{aligned} \quad (1)$$

We have set $\hbar = 1$ and written

$$\begin{aligned} [A, B]_+ &\equiv [A, B] \equiv AB - BA \\ [A, B]_- &\equiv \{A, B\} \equiv AB + BA , \end{aligned} \quad (2)$$

for the commutator and anti-commutator respectively, so that in Eq. (1) the sign (+) corresponds to bosons and (−) to fermions. The quantum number $m = 0, \pm 1, \pm 2, \dots$ indicates the angular momentum along the z -axis (which we

¹One of the authors (AJ) thanks his student Carlos Díaz for help understanding the treatment of this particular subject in the mechanical engineering literature (see, e.g., [15]), where the analogy with superradiance has not been noted. A thermodynamic approach may help to generalize and simplify such analyses.

²For historical accounts of this, see [17, 18].

take to be an axis of symmetry), $\omega \geq 0$ the energy, and α the spin together with any other quantum number needed to specify the field's state.

For the free field, the Hamiltonian is

$$H_f = \sum_{\omega, m, \alpha} \omega a_{m\alpha}^\dagger(\omega) a_{m\alpha}(\omega) \quad (3)$$

and the z -component of the angular momentum is

$$L_f^z = \sum_{\omega, m, \alpha} m a_{m\alpha}^\dagger(\omega) a_{m\alpha}(\omega) . \quad (4)$$

The most general field-bath interaction that is linear in the field operators is given by an additional term in the Hamiltonian of the form

$$H_{\text{int}} = \sum_{\omega, m, \alpha} (a_{m\alpha}(\omega) \otimes B_{m\alpha}^\dagger(\omega) + a_{m\alpha}^\dagger(\omega) \otimes B_{m\alpha}(\omega)) , \quad (5)$$

where $B_{m\alpha}(\omega)$ is a suitable bath operator. The bath has its own Hamiltonian H_b and its own z -component of the angular momentum L_b^z . The linearity of Eq. (5) with respect to the quantum field is a valid approximation as long as the field is sufficiently weak that its self-coupling (either induced by the medium's polarizability or direct as in gravity and non-Abelian gauge fields) can be neglected. All our computations will be in this weak-field regime, though we shall return to the issue of non-linearity in Sec. 9.

Assuming that the interaction described by Eq. (5) is invariant under rotations with respect to the z -axis, the bath operators may be chosen so that

$$[L_b^z, B_{m\alpha}(\omega)] = -m B_{m\alpha}(\omega), \quad [L_b^z, B_{m\alpha}^\dagger(\omega)] = m B_{m\alpha}^\dagger(\omega) , \quad (6)$$

and therefore

$$\begin{aligned} e^{i\phi L_b^z} B_{m\alpha}(\omega) e^{-i\phi L_b^z} &= e^{im\phi} B_{m\alpha}(\omega) , \\ e^{i\phi L_b^z} B_{m\alpha}^\dagger(\omega) e^{-i\phi L_b^z} &= e^{-im\phi} B_{m\alpha}^\dagger(\omega) . \end{aligned} \quad (7)$$

To take into account the bath's rotation we use an effective Hamiltonian of the form

$$H_b^{\text{eff}} = H_b - \Omega L_b^z , \quad (8)$$

where the internal H_b is independent of the rotation or, at most, parametrically dependent on Ω . The arbitrary sign in front of the Ω in Eq. (8) is taken negative for later notational convenience.

Equation (8) is a valid approximation as long as the bath's coherent rotation does not involve energies sufficiently large to excite internal degrees of freedom. This is analogous to the Born-Oppenheimer approximation in molecular physics, in which the electronic degrees of freedom are taken to be decoupled from the molecule's rotational and vibrational degrees of freedom [22]. We show in Sec. 7 that the kinetic energy of rotation can be converted into internal heating of the bath, but in our approximation this happens only via the coupling to the quantum field's low-energy modes.

Note that the Hamiltonian for our model is not necessarily quadratic, because $B_{m\alpha}(\omega)$ in Eq. (5) is arbitrary. The field's Markovian master equation that we derive for the weak-field limit in Sec. 4 applies, therefore, to models that are not exactly solvable. Even for exactly solvable models, our techniques help us to understand and to characterize the relevant thermodynamics.

3 Coupling spectrum and KMS condition

In the weakly coupled regime, all of the physically relevant properties of the bath are encoded in the second-order correlations of the bath operators $B_{m\alpha}(\omega)$, $B_{m\alpha}^\dagger(\omega)$, evaluated for the bath's stationary state [19, 23]. Using a short-hand notation B_k, B_l^\dagger with multi-index $k \equiv \{\omega, m, \alpha\}$, these correlations are expressed as matrix elements of the coupling spectrum

$$\gamma_{kl}^0(x) = \int_{-\infty}^{\infty} e^{ixt} \langle B_k^\dagger(t) B_l \rangle_b dt, \quad (9)$$

where $\langle \dots \rangle_b$ denotes the expectation value with respect to the given stationary state of the bath, while $B_k(t)$ is the bath observable B_k as it evolves in the Heisenberg picture for a bath governed by its internal Hamiltonian H_b . The matrix element $\gamma_{kl}^0(x)$ describes dissipative effects for frequency x , so that Eq. (9) is an instance of the fluctuation-dissipation theorem [24].

For the non-rotating bath in an equilibrium state, the following Kubo-Martin-Schwinger (KMS) condition is satisfied

$$\gamma_{kl}^0(-x) = e^{-\beta x} \gamma_{-l-k}^0(x), \quad \text{for } B_{-k} \equiv B_k^\dagger, \quad (10)$$

where β is the inverse temperature of the stationary bath and $-k \equiv \{\omega, -m, T\alpha\}$, with $T\alpha$ the time-reversal of α . Equation (10) relates the rates of decay and its time-reverse (i.e., pumping) for a field coupled to the stationary bath. [23, 25]

The coupling spectrum matrix is diagonal in m because of rotational symmetry. The weak coupling approach eliminates from the Markovian master equation (discussed in the next section) the non-diagonal elements in ω (the ‘‘secular approximation’’). Finally, the non-diagonal elements in α can be removed by a suitable choice of the modes. For simplicity, we use the same notation for this diagonal basis as for the original modes.

Replacing in Eq. (9) the internal Hamiltonian by the effective one of Eq. (8) and applying rotational invariance (see Eq. (7)) we obtain a modified diagonal coupling spectrum for the rotating bath

$$\gamma_{kk}^\Omega(x) = \gamma_{kk}^0(x + m\Omega). \quad (11)$$

This coupling spectrum satisfies a modification of the KMS condition of Eq. (10), namely

$$\gamma_{kk}^\Omega(-x) = e^{-\beta(x-m\Omega)} \gamma_{-k-k}^\Omega(x). \quad (12)$$

4 Markovian master equation

Using Davies's weak coupling limit technique [19], one may derive the Markovian master equation (MME) for the density matrix $\rho(t)$ that describes the quantum field and acts on the corresponding Fock space; see [23] for details. For the model in Sec. 2 we obtain that

$$\begin{aligned} \dot{\rho}(t) = & -i [H_f, \rho(t)] + \mathcal{L}\rho(t) = -i [H_f, \rho(t)] \\ & + \frac{1}{2} \sum_{m,\alpha,\omega} \Gamma_{m\alpha}^\Omega(\omega) \left\{ ([a_{m\alpha}(\omega)\rho(t), a_{m\alpha}^\dagger(\omega)] + [a_{m\alpha}(\omega), \rho(t)a_{m\alpha}^\dagger(\omega)]) \right. \\ & \left. + e^{-\beta(\omega-m\Omega)} ([a_{m\alpha}^\dagger(\omega)\rho(t), a_{m\alpha}(\omega)] + [a_{m\alpha}^\dagger(\omega), \rho(t)a_{m\alpha}(\omega)]) \right\}. \quad (13) \end{aligned}$$

Here \mathcal{L} is short-hand notation for the dissipative part of the Lindblad superoperator, which induces non-unitarity in the time evolution of the quantum state described by $\rho(t)$ [20, 21]. For convenience, we have used the same symbol H_f for the renormalized Hamiltonian in Eq. (13) —which contains corrections due to the field-bath interaction— as for the bare Hamiltonian in Eq. (3). The symbol $\Gamma_{m\alpha}^\Omega(\omega)$ in Eq. (13) denotes the diagonal element of the coupling spectrum matrix of Eq. (11) with $k = \{\omega, m, \alpha\}$, evaluated at frequency $x = \omega$.

The quantity

$$\gamma_\downarrow(k) \equiv \Gamma_{m\alpha}^\Omega(\omega) = \gamma_{kk}^0(\omega + m\Omega) \quad (14)$$

is the decay rate for the mode k , whereas

$$\gamma_\uparrow(k) \equiv e^{-\beta(\omega - m\Omega)} \Gamma_{m\alpha}^\Omega(\omega) = e^{-\beta(\omega - m\Omega)} \gamma_\downarrow(k) \quad (15)$$

is the corresponding pumping rate. The ratio of the pumping to the damping rates in Eq. (13) may be expressed as a Boltzmann factor with “local” (i.e., ω -dependent) inverse temperature $\beta_{\text{loc}}[\omega]$, such that

$$e^{-\beta_{\text{loc}}[\omega]} \equiv \frac{\gamma_\uparrow(k)}{\gamma_\downarrow(k)} = e^{-\beta(\omega - m\Omega)}, \quad (16)$$

implying that

$$\beta_{\text{loc}}[\omega] = \beta \left(1 - \frac{m\Omega}{\omega} \right). \quad (17)$$

Thus,

$$\beta_{\text{loc}}[\omega] < 0 \quad \Leftrightarrow \quad \omega < m\Omega. \quad (18)$$

Such a negative local temperature for the rotating bath indicates a population inversion in the low-energy modes with $\omega < m\Omega$. We shall see in Sec. 5 that this leads to a superradiant instability for bosonic fields (in which the Hamiltonian of a single mode is unbounded). In quantum optics it is common to describe the population inversion of a lasing medium in terms of its negative local temperature (see [26] for a general theory).

The modes with $\beta_{\text{loc}}[\omega] < 0$ are those for which the energy ω' measured in the bath’s comoving frame of reference is negative [1]. It is not surprising, therefore, that their equilibrium populations should be inverted, but our treatment clarifies how this depends on the field-bath interaction: note that we had to assume rotational invariance of the interaction in Eq. (5), and a large energy-scale separation between the bath’s rotational and internal degrees of freedom (Eq. (8)). We shall see that a thermodynamically complete description of superradiance must incorporate a *positive feedback* between field and bath.

5 Stable and unstable modes

The average occupation number of a single mode

$$\bar{n}_{m\alpha}(\omega, t) \equiv \text{Tr} [\rho(t) a_{m\alpha}^\dagger(\omega) a_{m\alpha}(\omega)] \quad (19)$$

obeys the equation

$$\begin{aligned} \dot{\bar{n}}_{m\alpha}(\omega, t) = & - \left(\Gamma_{m\alpha}^\Omega(\omega) \left[1 - (\pm) e^{-\beta(\omega - m\Omega)} \right] \right) \bar{n}_{m\alpha}(\omega, t) \\ & + \Gamma_{m\alpha}^\Omega(\omega) e^{-\beta(\omega - m\Omega)}, \end{aligned} \quad (20)$$

where, as in Eqs. (1) and (2), the sign (+) corresponds to bosons and (−) to fermions.

The fermionic solution to Eq. (20) for $t \geq 0$ is

$$\begin{aligned} \bar{n}_{m\alpha}(\omega, t) &= \exp \left\{ -\Gamma_{m\alpha}^{\Omega}(\omega) \left[1 + e^{-\beta(\omega - m\Omega)} \right] t \right\} \bar{n}_{m\alpha}(\omega, 0) \\ &+ \left[1 - \exp \left\{ -\Gamma_{m\alpha}^{\Omega}(\omega) \left[1 + e^{-\beta(\omega - m\Omega)} \right] t \right\} \right] \bar{n}_{m\alpha}(\omega, \infty) , \end{aligned} \quad (21)$$

with asymptotic population

$$\bar{n}_{m\alpha}(\omega, \infty) = \left[e^{\beta(\omega - m\Omega)} + 1 \right]^{-1} \quad (22)$$

corresponding to the Fermi-Dirac distribution with inverse temperature β and chemical potential $m\Omega$.

For bosons, on the other hand, only modes $|\omega, m, \alpha\rangle$ that satisfy

$$\omega > m\Omega \quad (23)$$

are stable. In that case the solution is

$$\begin{aligned} \bar{n}_{m\alpha}(\omega, t) &= \exp \left\{ -\Gamma_{m\alpha}^{\Omega}(\omega) \left[1 - e^{-\beta(\omega - m\Omega)} \right] t \right\} \bar{n}_{m\alpha}(\omega, 0) \\ &+ \left[1 - \exp \left\{ -\Gamma_{m\alpha}^{\Omega}(\omega) \left[1 - e^{-\beta(\omega - m\Omega)} \right] t \right\} \right] \bar{n}_{m\alpha}(\omega, \infty) , \end{aligned} \quad (24)$$

with asymptotic population

$$\bar{n}_{m\alpha}(\omega, \infty) = \left[e^{\beta(\omega - m\Omega)} - 1 \right]^{-1} , \quad (25)$$

corresponding to the Bose-Einstein distribution with inverse temperature β and chemical potential $m\Omega$. Modes satisfying the condition

$$\omega < m\Omega , \quad (26)$$

on the other hand, are unstable and their occupation numbers grow exponentially with time. This corresponds to Zel'dovich's rotational superradiance.

It is interesting to consider the case of zero temperature, i.e., the limit $\beta \rightarrow \infty$ in Eq. (20) for superradiant modes. Using the KMS conditions of Eqs. (10) and (12), one obtains for the pumping rate

$$\gamma_{\uparrow}(k) = \Gamma_{m\alpha}^{\Omega}(\omega) e^{-\beta(\omega - m\Omega)} = \gamma_{-k-k}^0(m\Omega - \omega) \equiv \gamma_{\omega m\alpha}(m\Omega - \omega) , \quad (27)$$

where $\gamma_{\omega m\alpha}(x)$, for $x \geq 0$ is the damping rate of the mode $|\omega, -m, \alpha\rangle$ at the frequency x . Note that this damping rate remains positive in the zero-temperature limit, while $\gamma_{\omega m\alpha}(-x)$ tends to zero. Thus, the rate in Eq. (20), in the zero-temperature limit, becomes

$$\dot{\bar{n}}_{m\alpha}(\omega, t) = \gamma_{\omega m\alpha}(m\Omega - \omega) [1 + \bar{n}_{m\alpha}(\omega, t)] . \quad (28)$$

Equation (28) implies that a rotating body at zero temperature will produce a continuous spectrum of radiation, with a non-trivial spatial distribution determined by the superradiant condition of Eq. (26).

6 Feedback and stimulated emission

The structure of Eq. (13) implies that the different field modes $|\omega, m, \alpha\rangle$ evolve independently. Moreover, the diagonal matrix elements of $\rho(t)$, computed in the corresponding population number representation basis, give probabilities $P_n(k; t)$ of finding n particles in a given mode $|k\rangle \equiv |\omega, m, \alpha\rangle$. These probabilities evolve according to a Markovian birth-death process

$$\begin{aligned} \dot{P}_n(k; t) = & \gamma_\downarrow(k)(n+1)P_{n+1}(k; t) + \gamma_\uparrow(k)[1 \pm (n-1)]P_{n-1}(k; t) \\ & - [\gamma_\downarrow(k)n + \gamma_\uparrow(k)(1 \pm n)]P_n(k; t) . \end{aligned} \quad (29)$$

In the symbol \pm on the right-hand side of Eq. (29), the sign $(+)$ corresponds to bosons (with unbounded population $n = 0, 1, 2, \dots$) and the sign $(-)$ to fermions (with bounded population $n = 0, 1$).

From the last term on the right-hand side of Eq. (29) we see that the probability, per unit time, of creating a new particle if n particles are already present in a given mode depends on n and is equal to $\gamma_\uparrow(k)(1 \pm n)$, where the 1 corresponds to spontaneous emission. This n -dependence can be interpreted as resulting from a feedback between the field and the bath. This feedback is positive for bosons, due to stimulated emission. For fermions the feedback is negative, due to the Pauli exclusion principle.

Superradiance is a process in which the kinetic energy of the heat bath's rotation powers coherent radiation modes, without requiring any resonant tuning. In a classical context, such processes are described as “self-oscillations”. For a review of self-oscillation as a non-equilibrium process depending on a positive feedback between the oscillator and the power source, see [27].

7 Energy and entropy balance

The formal Gibbs state

$$\bar{\rho} = Z^{-1} e^{-\beta(H_f - \Omega L_f^z)} \quad (30)$$

is a stationary solution of Eq. (13). This allows us to apply the general formula for the entropy production based on the Spohn inequality [28, 29],

$$\sigma(t) = -\text{Tr} [\mathcal{L}\rho(t) (\ln \rho(t) - \ln \bar{\rho})] \geq 0 , \quad (31)$$

in order to derive the entropy balance (i.e., the second law of thermodynamics). In units such that $k_B = 1$, this gives

$$\dot{S}(t) = \sigma(t) + \beta J , \quad (32)$$

where $S(t) \equiv -\text{Tr} [\rho(t) \ln \rho(t)]$ is the entropy of the field, while

$$J = \frac{d}{dt} \text{Tr} [\rho(t) (H_f - \Omega L_f^z)] . \quad (33)$$

is the heat current flowing from the bath and into the field. Identifying the internal energy U of the field with its average value of the Hamiltonian,

$$U(t) = \text{Tr} [\rho(t) H_f] , \quad (34)$$

we obtain the energy balance (i.e., the first law of thermodynamics)

$$\dot{U}(t) = J + \Omega \frac{d}{dt} \text{Tr} [\rho(t) L_{\text{f}}^z] , \quad (35)$$

where the second term on the right-hand side of Eq. (35) is the power supplied by the rotating bath.

Note that the Gibbs state of Eq. (30) cannot be normalized for bosonic modes with $\omega < m\Omega$ (i.e., for the superradiant modes). However, introducing intermediate cutoffs in particle numbers one can derive rigorously the results of Eqs. (32) and (35).

Combining Eq. (33) with the kinetic Eq. (28) provides a simple derivation of the fact that, at zero temperature, superradiance is accompanied by a heating of the bath. Indeed, the corresponding expression for the heat current into the field,

$$J = \sum_{\{\omega < m\Omega, m, \alpha\}} (\omega - m\Omega) \cdot \gamma_{\omega m \alpha} (m\Omega - \omega) \cdot [1 + \bar{n}_{m\alpha}(\omega, t)] , \quad (36)$$

is evidently negative.

In comparing our results to superradiant spectra computed by other authors for specific systems, one should bear in mind that we formulated the model of Sec. 2 in terms of the field modes as they would exist within a cavity containing the heat bath. The non-unitarity represented by the superoperator \mathcal{L} in Eq. (13) comes from the dynamical interaction with the bath, without any modification of the field's boundary conditions. Other treatments often frame superradiance as a scattering problem, with at least some of the non-unitarity given by the choice of boundary conditions within the bath; see, e.g., [30, 31].

8 Black hole radiation

For any quantum field placed around a BH, its modes can be separated into those localized far outside the BH (outer modes) and those close or below the event horizon (inner modes). The particles occupying the inner modes form a bath for the outer ones. The interaction between both systems can be treated as a kind of tunneling process described by the quadratic Hamiltonian of the form of Eq. (5), with the operator $a_{m\alpha}^\dagger(\omega)$ creating a particle in an outer mode and $B_{m\alpha}(\omega)$ annihilating a particle in a certain superposition of inner modes.

As shown by Hawking in [16], the strong gravity of a BH creates indeterminacy between the annihilation operator b_k and the creation operator b_k^\dagger of an inner mode with multi-index k . Hence, we can introduce a bath operator in Eq. (5) of the form

$$B_k = \sum_{k'=\{\omega', m, \alpha'\}} \left[f_k(\omega') b_{k'} + g_k(\omega') b_{-k'}^\dagger \right] \quad (37)$$

with some form factors $f_k(\omega), g_k(\omega)$ (as before, $-k$ stands for the time-reversal of the quantum numbers in k). There is no summation over $m' \neq m$ in Eq. (37) because we assumed rotational symmetry.

The key result of [16] is that the indeterminacy between the annihilation and creation operators is exponentially small and approximately given by

$$\frac{|g_k(\omega)|^2}{|f_k(\omega)|^2} = e^{-\beta_H \omega} \quad (38)$$

where β_H is the inverse Hawking temperature of the BH. Inserting Eqs. (37) and (38) into Eq. (9) for the vacuum state $|0\rangle$ of the inner modes (i.e. the state $|0\rangle$ such that $b_k|0\rangle = 0$ for all k 's), one obtains the KMS condition of Eq. (10) with $\beta = \beta_H$, thereby establishing that the static black hole behaves with respect to the external fields as a heat bath at the Hawking temperature.

The results of Sec. 5 then imply that a Kerr BH will superradiate bosons obeying the condition of Eq. (26), with a spectrum determined by the form factors in Eq. (37). The negative energy of a superradiant mode as measured in the Kerr BH's co-moving frame (see Sec. 4) reflects the presence of an ergosphere allowing extraction of the BH's angular momentum by the Penrose process [32].

In the thermodynamic relation for a Kerr-Newman BH with mass M , angular momentum L , charge Q , event horizon area A , and Hawking temperature T_H ,

$$dM = T_H \frac{dA}{4} + \Omega dL + \Phi dQ, \quad (39)$$

the angular velocity Ω and the electrostatic potential Φ near the event horizon appear as chemical potentials. (In Eq. (39) the fundamental constants c , \hbar , and G have all been set to 1.) Rotational and charged superradiance are processes that extract some of the BH's internal energy ($dM < 0$), while increasing the entropy within the BH's event horizon ($dA > 0$) [34]; see also [8] and references therein. The energy source is non-thermal, but the irreversibility of superradiance relates it to the purely thermal Hawking radiation of a BH with $\Omega = 0$ and $\Phi = 0$. Since Hawking radiation does not depend on stimulated emission, it produces fermions as well as bosons.³

The relation between BH superradiance and Hawking radiation is analogous to the relation between the coherent radiation produced by a laser and the thermal radiation that the unpumped lasing medium at non-zero temperature would emit into a surrounding vacuum (a radiation associated with the non-zero $\gamma_{\uparrow}(k)$'s). Though distinct, the two processes are tied by thermodynamic irreversibility.

9 Classical limit and flow instabilities

The Kelvin-Helmholtz (KH) instability in hydrodynamics, which explains how the wind makes waves on the surface of the ocean, is an interesting classical analog of the superradiance of bosonic fields.⁴ It is instructive to consider how the thermodynamic arguments of [2, 5, 7] apply to the KH instability, and to identify the hydrodynamical positive feedback as well as the reasons why it leads to a result similar to that given by stimulated emission of bosons.

³The absence of fermionic superradiance (about which we will have more to say in Sec. 9) appears, in the context of Hawking's area theorem [33], as tied to the fact that fermions violate the weak energy condition [34, 35].

⁴John McGreevy brought this to our attention some years ago.

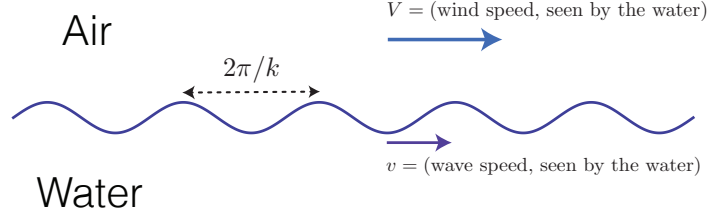


Figure 1: Illustration of the hydrodynamic Kelvin-Helmholtz (KH) instability by which wind can generate waves on the surface of a body of water. Image adapted from [36].

The KH instability can occur at the interface between two layers of fluid, as long as there is a difference in their respective tangential velocities. Let us consider an upper layer of air with mass m , which moves with horizontal velocity V with respect to the water below, as shown in Fig. 1. Non-relativistically, the air's momentum is $\mathbf{p}_{\text{wind}} = m\mathbf{V}$ and its corresponding kinetic energy is

$$E_{\text{wind}} = \frac{p^2}{2m} . \quad (40)$$

The air's viscosity provides a coupling between the air and the surface of the water. Consider a traveling wave on the water's surface, which in the linear regime may be described by a complex-valued

$$\xi \sim e^{i(kx - \omega t)} , \quad (41)$$

where x is the spatial coordinate in the horizontal direction, k the wavenumber, and

$$v = \frac{\omega}{k} \quad (42)$$

the phase velocity. Translational invariance implies momentum conservation

$$\dot{\mathbf{p}}_{\text{wind}} = -\dot{\mathbf{p}}_{\text{wave}} . \quad (43)$$

We may express the rate at which the wave is gaining momentum as

$$\dot{\mathbf{p}}_{\text{wave}} = f\hbar\mathbf{k} , \quad (44)$$

where $\hbar\mathbf{k}$ is the momentum of a quantum of wave and f is the rate at which such quanta are being produced (for clarity, in this section we write the factors of \hbar explicitly). Taking the time derivative of Eq. (40) and using Eqs. (43) and (44) gives

$$-\dot{E}_{\text{wind}} = \frac{\mathbf{p}}{m} \cdot \dot{\mathbf{p}}_{\text{wave}} = \mathbf{V} \cdot (f\hbar\mathbf{k}) \quad (45)$$

Since each wave quantum carries energy $\hbar\omega$, we also have that

$$\dot{E}_{\text{wave}} = f\hbar\omega = f\hbar vk \quad (46)$$

Combining Eqs. (45) and (46) we conclude that

$$\left| \dot{E}_{\text{wind}} \right| > \dot{E}_{\text{wave}} \Leftrightarrow V > v . \quad (47)$$

The condition for the KH instability is

$$V > v . \quad (48)$$

From Eq. (47) we can see that this is because, in such a case, dissipative processes can, while conserving momentum, extract more kinetic energy from one fluid layer (in our case, from the wind) than goes into amplifying waves on the interface. That energy difference is available to generate entropy (here by viscous dissipation in the air) and the process therefore proceeds irreversibly. This thermodynamic analysis is trivially generalizable to systems with a discontinuity in rotational rather than translational velocity and to relativistic systems, as in [2, 5, 7].

Zel'dovich noted in [1] that quantization makes the argument particularly simple and concluded that this was another instance of how “quantum mechanics helps understand classical mechanics”.⁵ Only the bosonic case, with no upper bound on the number of quanta with wavenumber k , can approach a classical regime. It is interesting to consider how the feedback from stimulated emission, described in Sec. 6, appears in the classical flow instability.

Classical self-oscillation is due to a backreacting force that *leads* the oscillation and therefore amplifies it [27]. In the KH instability this results from the bath moving faster than the wave’s phase, which is equivalent to negativity of the energy

$$\omega' = \frac{i}{\xi} \frac{\partial \xi}{\partial t'} = \frac{i}{\xi} \left(\frac{\partial \xi}{\partial t} - V \frac{\partial \xi}{\partial x} \right) = \omega \left(1 - \frac{V}{v} \right) \quad (49)$$

in the bath’s comoving frame (as it does not affect our reasoning, we omit the relativistic factor $(1 - V^2/c^2)^{-1/2}$ in Eq. (49)). If $V > v$, then $\omega' < 0$ and the oscillation sees the time lag resulting from intra-bath dissipation as a lead, thus establishing a positive feedback between the wave and the air pressure above it.

In the equation of motion for a scalar ξ , the condition $\omega' < 0$ also implies that the sign of the linear damping term $a(\partial \xi / \partial t)$ flips upon transformation to the bath’s comoving frame (this was Zel’dovich’s first argument for superradiance in [1]). A fermionic field’s damping appears in its equation of motion as an imaginary part of the invariant mass, which therefore does not change sign for $\omega' < 0$. Absence of fermionic superradiance may also be seen from the solutions to scattering off a potential barrier [30] or, in an AdS/CFT context, from the behavior of the poles of the retarded Green functions corresponding to boundary-theory operators [38].

Hydrodynamically, one may see the KH instability of Fig. 1 as a consequence of the fact that, when $v < V$, the air “trapped” in a valley of the wave is slowed down, causing the air pressure there to increase (by Bernoulli’s theorem) relative to the crests. This, combined with the lag of that air pressure oscillation with respect to the phase of the wave—a lag resulting from the air’s viscosity and

⁵He had previously published a brief, pseudonymous piece with that title [37].

therefore associated with dissipation in the air— implies that the air pressure does net positive work on the wave over a full period.⁶

Linear feedback causes the wave’s amplitude to grow exponentially with time. Non-linear dissipative effects eventually limit the wave’s growth, giving a steady amplitude that corresponds to a limit cycle in the classical phase space [27]. The non-linear approach to a limit cycle is evidently an irreversible process that erases information about initial conditions and transient noise. The initial, linear runaway is also irreversible, but to see that we must take into account the dynamics of the bath (the wind in Fig. 1) in which entropy is generated.

To better understand the quantum-classical correspondence for such systems, consider the bosonic birth-death process described by Eq. (29), but with the probability of single-boson absorption per unit time replaced by a more general expression

$$\gamma_{\downarrow}(k)n \rightarrow \gamma_{\downarrow}(k)(n + \kappa n^2 + \dots) \text{ , for } \kappa \geq 0 \text{ .} \quad (50)$$

This corresponds to going beyond the linear coupling of Eq. (5), from which we derived the simplest MME (Eq. (13)). A $\kappa > 0$ in Eq. (50) can, for instance, describe gain saturation in a laser [39].

Denote by $\langle \dots \rangle$ the average of an observable over the probabilities $P_n(k; t)$. For $\kappa > 0$ there is no closed kinetic equation for the average population $\langle n \rangle$ of a given mode, because of the presence of the higher-order term $\langle n^2 \rangle$. However, in the semi-classical limit the fluctuations of the population numbers may be neglected, so that one can replace $\langle n^2 \rangle$ by $\langle n \rangle^2$. This gives a kinetic equation for $\langle n \rangle$ that can be interpreted classically, with the quantum stimulated emission transformed into a non-linear feedback that gives rise to a limit cycle. This is an instance of non-linear classical mechanics emerging from the linear evolution of quantum states.

The general relation between the kinetic equations derived from the Markovian birth-death process for the quantum system and the hydrodynamical description of the system lies beyond the scope of this paper. But the argument presented here illustrates how the large- n limit of bosonic superradiance is connected to a hydrodynamic instability.

10 Discussion

Superradiance is by now a well-known effect in high-energy physics, general relativity, and quantum optics. The contribution of this article was to derive it in the formalism of the MME for the quantum field coupled to a heat bath with a rotational symmetry that admits an effective description of the form of Eq. (8). Rather than assuming the laws of thermodynamics, we have derived them (see Sec. 7) from the dynamics of the field as an open quantum system. This clarified how superradiance depends on the feedback between the field and the bath, and on the generation of entropy in the bath.

⁶That an inviscid wind does no work on the wave is nicely underlined and illustrated in [40]. Textbook treatments (see, e.g., [14]) find a KH instability for inviscid flow because they consider infinitesimal perturbations for which both the power transferred to the wave (\dot{E}_{wave}) and the power dissipated in the air ($|\dot{E}_{\text{wind}}| - \dot{E}_{\text{wave}}$) vanish. Quantization avoids this unphysical limit.

This approach also allowed us to fix the possible dependence of the superradiant spectrum on the rotational velocity Ω , which appears exclusively through the frequency shift of Eq. (11). In Sec. 9 we clarified the connection between superradiance as a quantum process that proceeds via stimulated emission, and classical self-oscillations such as flow instabilities. Our results sharpen the arguments of [8, 31] that superradiance requires not just the presence of negative-energy states (associated, in a Kerr BH geometry, to the ergosphere), but also non-unitarity in the field’s time evolution (given, in a BH geometry, by the event horizon). We saw that in the weak-field, linear-response approximation this non-unitarity, represented by the \mathcal{L} in the MME of Eq. (13), has a clear and consistent thermodynamic interpretation. In Sec. 9 we commented on how non-linearity is related to gain saturation and to the approach to a classical limit cycle, but a detailed understanding of this non-linear regime remains a challenge to be met by future research.

The thermodynamic point of view allows us to abstract much of the detailed physics and to treat many distinct systems within a simple and unified framework. The wisdom of this approach has been demonstrated in recent decades by the fruitfulness of BH thermodynamics [41].⁷ Incorporating into this thermodynamical picture a more sophisticated understanding of the evolution of open quantum systems may shed light on fundamental problems.

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⁷Thorne’s account of Zel’dovich’s original argument for BH superradiance in [18] brings to mind an aphorism that one of us (AJ) received as a student from Gil Refael: “Thermodynamics is saying something without knowing anything.”

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